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ORDER, DISORDER AND GENERALIZED STATISTICS

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We generalize the prescription of Kadanoff and Ceva for the computation of disorder variable correlation functions in the Ising model for continuous field theories with U(1) symmetry. By considering the product of order and disorder variables, we obtain a path integral representation for fields with generalized statistics. We discuss in detail the cases of massless Thirring and Schwinger models.

1. Introduction

It is a property of two-dimensional world that the product of bosonic fields may result in a new field with generalized statistics ("spin") not necessarily Bose or Fermi. These fields with unconventional statistics play a central role in models like the chiral Gross-Neveu [1, 2], $Z(N)$ in the scaling limit [3] and allow for a generalization of the Schwinger and Thirring models [4].

In four dimensions, the analogous feature is the possibility of building a fermionic field as a product of bosonic ones, related by duality relation [5], so that the dyon build-up from scalar electric and magnetic monopole fields behaves as a fermion [6].

The functional integral formulation of a field theory is only known for fermion or boson fields. It is the purpose of this work to formulate the functional integral directly in terms of generalized statistics fields.

To do this, we make use of the statistical mechanics concepts of order and disorder and observe that at the classical statistical mechanics level, a bosonized field in the Mandelstam form [7] may be viewed as a product of order and disorder variables.

Kadanoff and Ceva [8] have showed how to compute correlation functions of such products of order and disorder variables, σ and μ, in the two-dimensional Ising model, where it is known that a fermion has the structure $\sigma\mu$ [9].

The method consists in modifying the couplings (ferromagnetic→antiferromagnetic) along a path joining the disorder variables. Due to the symmetry of the partition function, it is proved that this procedure is path independent.

We generalize their method to a class of continuous field theories and arrive in this way at a direct bosonization of the functional integral* which allows its formulation for generalized statistics fields.

In sect. 2, we apply the method to the massless Thirring model and obtain the Schwinger correlation functions for general spin. We also prove the equivalence of this model with a two-dimensional electrostatic system composed of electric charges and strings of electric dipoles ("magnetic monopoles"). The correlation functions are the exponentials of the interaction energy of this system and the ψ field may be considered as a bound-state charge-"monopole" (dyon). Since the charges of this electrostatic system correspond to the pseudocharge and the "magnetic-monopoles" correspond to the actual charge, it is convenient to modify one's language, calling charges what we had called "magnetic-monopoles" and *vice-versa*.

The selection rules of the model appear in a very natural way in this formulation.

We show that the euclidean space has a many-sheeted structure determined by the spin, and associate the various orderings of the correlation functions with this structure.

We show that our procedure is path independent, as in ref. [8]. In the massive Thirring model, path independence implies quantization of the "magnetic monopoles" (actual charge).

In sect. 3, we apply the method to the massless Schwinger model (QED_2). In this case the functional integral is path dependent, leading naturally to the gauge invariant correlation functions both in the θ and n vacua [10].

We prove that the massless Schwinger model is equivalent to a two-dimensional magnetostatic system of magnetic charges and strings of magnetic dipoles (actual electric monopoles) embedded in a magnetic plasma. Again the gauge invariant correlation functions are the exponentials of the interaction energy of this system, and the ψ field may be considered as a bound-state charge-monopole. The effect of the plasma is to make the string physical, that is, a tube of electric flux, thus confining the actual charges. We may, therefore, understand the confinement of the Schwinger model as a condensation of magnetic monopoles in the vacuum, in close analogy to the standard picture of 4-dimensional confinement [11]. This condensate confines the actual charges.

All features involving many-sheetedness and ordering that appeared in the Thirring model are also manifest in the Schwinger model.

By reinterpreting the functional integral, and not considering the cuts along the strings, we also obtain the gauge dependent correlation functions in the Landau gauge. The functional integral is path independent in this case.

* After the completion of this work, our attention was called by R. Köberle to a paper by Zamolodchikov [23], where some of our results on the Thirring model are already contained. There are also some related results within the statistical mechanics framework [25].

Our method also allows for a generalization of the Schwinger model for arbitrary spin.

In sect. 4, we summarize our results.

2. Thirring model

We may write the massless Thirring field in the form [7] (we will not worry about constant multiplicative factors that are present in the bosonized form of ψ):

$$\psi(x) = \exp\left\{ia\gamma^5\phi(x) - ib\int_{x,C}^{\infty} \varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,dz_\mu\right\}\binom{1}{1}, \tag{2.1}$$

where ϕ is a free massless bosonic field (the pseudo-potential), the integral is taken along an arbitrary path C and a and b are constants determining the spin of ψ, $s = ab/2\pi$ [12].

Let us call

$$\sigma(x) = \exp\{ia\gamma^5\phi(x)\}, \qquad \mu(x) = \exp\left\{-ib\int_{x,C}^{\infty} \varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,dz_\mu\right\}. \tag{2.2}$$

Then, we have

$$\mu(t, y_1)\sigma(t, x_1) = \sigma(t, x_1)\mu(t, y_1)\exp\{i2\pi\gamma^5 s\theta(x_1 - y_1)\}, \tag{2.3}$$

that is, the commutation between σ and μ produces a dislocation in the ϕ field if σ is to the right of μ and leaves it unchanged otherwise. This is the analog of the commutation relation between order and disorder variables in the Ising model, if we note that the symmetry of that model is $\sigma \to -\sigma$ and of the Thirring model is $\phi \to \phi + K$.

Therefore, we may consider the ψ field of the Thirring model as a product of an order and a disorder variable and generalize the prescription of Kadanoff and Ceva for the computation of correlation functions of such objects [8].

Thus, we write, for the order-order correlation function, the euclidean functional integral

$$\langle\sigma(x)\bar{\sigma}(y)\rangle = N\int [D\phi]\exp\left\{\tfrac{1}{2}\int d^2z\phi\,\partial^2\phi\right\}\exp\{ia[\gamma_x^5\phi(x) + \gamma_y^5\phi(y)]\}, \tag{2.4}$$

where N is the usual normalization factor. We may put (2.4) in the form

$$\langle\sigma(x)\bar{\sigma}(y)\rangle = N\int [D\phi]\exp\left\{-\int d^2z[-\tfrac{1}{2}\phi\,\partial^2\phi + \phi(z)\alpha(z)]\right\}, \tag{2.5}$$

where

$$\alpha(z) = -ia[\gamma_x^5\delta(z - x) + \gamma_y^5\delta(z - y)]. \tag{2.6}$$

The integral (2.5) is saturated by the solutions of the Poisson equation $\nabla^2\phi = \alpha$. To solve it, we impose the condition that ϕ vanishes on a circle of radius R and then take R going to infinity. The result is straightforward, giving

$$\langle\sigma(x)\bar{\sigma}(y)\rangle = \exp\left\{\tfrac{1}{2}\int \mathrm{d}^2 z\alpha(z)\phi_\alpha(z)\right\}, \tag{2.7}$$

with

$$\phi_\alpha(z) = -ia[\gamma_x^5 D(z-x) + \gamma_y^5 D(z-y)] + iaD(\infty)[\gamma_x^5 + \gamma_y^5], \tag{2.8}$$

where $D(z) = -(1/2\pi)\ln|z|$ is the massless Green function, satisfying $\partial^2 D(z) = -\delta(z)$.

Eq. (2.7) with (2.6) and (2.8) is the exponential of the interaction energy of an electrostatic system of two charges placed at x and y.

Evaluating (2.7), and neglecting the classical self-energy terms, we arrive directly at the renormalized correlation functions

$$\langle\sigma_\mathrm{R}(x)\bar{\sigma}_\mathrm{R}(y)\rangle = \exp\left\{\frac{a^2}{2\pi}\gamma_x^5\gamma_y^5 \ln|y-x| + a^2 D(\infty)[1+\gamma_x^5\gamma_y^5]\right\}. \tag{2.9}$$

The elements (1, 1) and (2, 2), vanish, because of the last term in (2.9), leading to the chiral selection rule. For the non-vanishing components (1, 2) and (2, 1), we get

$$\langle\sigma_\mathrm{R}(x)\bar{\sigma}_\mathrm{R}(y)\rangle_{12} = \langle\sigma_\mathrm{R}(x)\bar{\sigma}_\mathrm{R}(y)\rangle_{21} = |y-x|^{-a^2/2\pi}. \tag{2.10}$$

This correlation functions are the continuous limit of the low-temperature regime of the $X-Y$ model [13].

In the same way as in (2.4), we write for the disorder-disorder correlation function

$$\langle\mu(x)\bar{\mu}(y)\rangle = N\int [\mathrm{D}\phi]\exp\left\{\tfrac{1}{2}\int \mathrm{d}^2 z\phi\,\partial^2\phi\right\}\exp\left\{b\int_{x,C}^{y} \varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,\mathrm{d}z_\mu\right\}, \tag{2.11}$$

where C is an, arbitrary path joining x and y and the exponential of the line integral is real since we are working in the euclidean region.

We may put (2.11) in the form

$$\langle\mu(x)\bar{\mu}(y)\rangle = N\int [\mathrm{D}\phi]\exp\left\{-\int \mathrm{d}^2 z[-\tfrac{1}{2}\phi\,\partial^2\phi + \phi(z)\beta(z)]\right\}, \tag{2.12}$$

with

$$\beta(z) = -b\int_{x,C}^{y} \varepsilon^{\mu\nu}\,\partial_\nu^{\mathfrak{z}}\delta(z-\mathfrak{z})\,\mathrm{d}\mathfrak{z}_\mu. \tag{2.13}$$

Let us show that (2.12) is path independent. To do this, observe that

$$b\int_{x,C}^{y} \varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,\mathrm{d}z_\mu = b\int_{x,C'}^{y} \varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,\mathrm{d}z_\mu + b\int_\mathrm{S} \partial^2\phi\,\mathrm{d}^2 z, \tag{2.14}$$

where S is the region closed by C and C′. Inserting (2.14) in (2.11) and making a change of variable $\phi \to \phi + b$ within the region S, we obtain the same expression with C′ instead of C. The boundary divergent term which eventually could arise can be incorporated in the renormalization factors of μ and $\bar{\mu}$.

Again (2.12) is saturated by the solutions of the Poisson equation with the source, β, given by a string of electric dipoles.

Computing (2.12), we obtain

$$\langle \mu(x)\bar{\mu}(y)\rangle = \exp\left\{\tfrac{1}{2}\int \mathrm{d}^2 z \beta(z)\phi_\beta(z)\right\}, \tag{2.15}$$

where

$$\phi_\beta(z) = \frac{b}{2\pi}[\arg(z-x) - \arg(z-y)]. \tag{2.16}$$

Evaluating the integral in (2.15) we arrive at

$$\langle \mu_{\mathrm{R}}(x)\bar{\mu}_{\mathrm{R}}(y)\rangle = \exp\left\{-\frac{b^2}{2\pi}\ln|y-x|\right\} = |y-x|^{-b^2/2\pi}. \tag{2.17}$$

In the various steps leading to (2.17) we have used the Cauchy–Riemann equations

$$\varepsilon^{\mu\nu}\partial_\nu[\arg(z-x) - \arg(z-y)] = \partial^\mu[\ln|z-x| - \ln|z-y|] - 2\pi\int_{x,\mathrm{C}}^{y} \delta(z-\xi)\,\mathrm{d}\xi_\mu\,, \tag{2.18}$$

and neglected the terms corresponding to the self-energy of the string and of the "monopoles" at its ends, getting in this way the renormalized disorder correlation functions.

Eq. (2.17) is the exponential of the interaction energy of a "monopole"–"anti-monopole" pair placed respectively at x and y. Note that the fact that while charges in the computation of the order correlation function are pure imaginary, whereas the monopoles appearing in the disorder correlation functions are real, is trivially related to the fact that opposite charges attract each other and opposite currents repel.

Since the $\langle\mu\rangle$ correlation function vanishes because of the long-range behaviour of the monopole field, one obtains the charge selection rule of the Thirring model.

To arrive at the ψ field correlation functions, we consider now the four-point order–disorder Schwinger function

$$\langle \psi_{\mathrm{R}}(x)\bar{\psi}_{\mathrm{R}}(y)\rangle = \lim_{\substack{x_1, x_2 \to x \\ y_1, y_2 \to y}} \langle \sigma(x_1)\mu(x_2)\bar{\sigma}(y_1)\bar{\mu}(y_2)\rangle\,. \tag{2.19}$$

In the same way, we obtain for (2.19)

$$\langle\psi_{\mathrm{R}}(x)\bar{\psi}_{\mathrm{R}}(y)\rangle=\lim_{\substack{x_1,x_2\to x\\ y_1,y_2\to y}}\exp\left\{\frac{a^2}{2\pi}\gamma_x^5\gamma_y^5\ln|y_1-x_1|-\frac{b^2}{2\pi}\ln|y_2-x_2|\right.$$
$$+\frac{iab}{2\pi}[\gamma_x^5\arg(x_1-y_2)-\gamma_y^5\arg(y_1-x_2)-\gamma_x^5\arg(x_1-x_2)$$
$$\left.+\gamma_y^5\arg(y_1-y_2)]+a^2D(\infty)[1+\gamma_x^5\gamma_y^5]\right\}. \tag{2.20}$$

The above expression now corresponds to the exponential of the interaction energy of a system of imaginary charges and real "monopoles". The selection rules obtained before are also present here.

Notice that the mixed four-point function is now multisheeted, and path independence is valid only if charges do not cross strings, as in the analogous Ising problem [8, 9]. This fact will reflect itself in the ambiguity related to the various orderings of correlation functions.

When we take the limit in (2.20), there appears an ambiguity arg (0) + arg (0). This ambiguity is related, to a direction-dependent renormalization factor already present in the operator formulation of the theory [14]. We overcome it by taking the limit in opposite angles.

A second ambiguity of the form $e^{im2\pi s}$, $m=0,\pm1,\pm2,\dots$, arises depending on the number of times, m, we cross the string when we take the limit $x_1\to x_2=x$ and $y_1\to y_2=y$. This ambiguity is intrinsic of our formulation, and as we shall see, highly desirable.

Then, after taking the limit we obtain, for the (1, 2) component, for instance,

$$\langle\psi_{\mathrm{R}}(x)\bar{\psi}_{\mathrm{R}}(y)\rangle_{12}$$
$$=e^{im2\pi s}\exp\{-[(a^2+b^2)/2\pi]\ln|y-x|-is[\arg(y-x)+\arg(x-y)]\}. \tag{2.21}$$

Eq. (2.21) is, except for the ambiguity factor, exactly the Schwinger function corresponding to the continuous spin, Klaiber solutions [4, 12].

The general correlation function for the ψ field would be obtained by considering the exponential of the interaction energy of a system with an arbitrary number of charges and "monopoles" [15].

Observe now that our functional integral is the same for $\langle\psi\bar{\psi}\rangle$ as well as for $\langle\bar{\psi}\psi\rangle$. These functions are in general different and the ambiguity factor in front of (2.21) gives us the various possibilities. It is a signal of the many sheetedness of the space, the number of sheets depending on the spin. For bosons there is only one sheet, for fermions two; for spin-$\frac{1}{3}$, for instance, there are three sheets and so on.

For fermions, we obtain $\langle\psi\bar{\psi}\rangle$ by taking the limit with the two charges in the same sheet ($m=0,\pm2,\dots$) and $\langle\bar{\psi}\psi\rangle$ is obtained with one of them in the first sheet and the other in the second one ($m=\pm1,\pm3,\dots$).

For unconventional spin, the Schwinger functions of the ψ field are themselves multivalued. We illustrate the $s=\frac{1}{3}$ case. Starting from $\langle\psi\bar{\psi}\rangle$ and changing the order in the Schwinger function, we get:

$$\begin{aligned}\langle\psi\bar{\psi}\rangle_1 &= e^{-i2\pi/3}\langle\bar{\psi}\psi\rangle_1 = e^{-i4\pi/3}\langle\psi\bar{\psi}\rangle_2 = e^{-i6\pi/3}\langle\bar{\psi}\psi\rangle_2 \\ &= e^{-8\pi/3}\langle\psi\bar{\psi}\rangle_3 = e^{-i10\pi/3}\langle\bar{\psi}\psi\rangle_3 = e^{-i12\pi/3}\langle\psi\bar{\psi}\rangle_1 = \cdots\,, \qquad (2.22)\end{aligned}$$

indicating that $\langle\psi\bar{\psi}\rangle$ and $\langle\bar{\psi}\psi\rangle$ have three sheets, differing each one by the factor $e^{i4\pi/3}$. The subscript indicates the sheet in consideration. $\langle\psi\bar{\psi}\rangle_1$ has the same value as $\langle\bar{\psi}\psi\rangle_2$, $\langle\psi\bar{\psi}\rangle_2$ has the same value as $\langle\bar{\psi}\psi\rangle_3$ and $\langle\psi\bar{\psi}\rangle_3$ has the same value as $\langle\bar{\psi}\psi\rangle_1$.

Taking the limit with the two charges in the same sheet of our euclidean space ($m=0, \pm 3, \ldots$), we obtain $\langle\psi\bar{\psi}\rangle_1$ or $\langle\bar{\psi}\psi\rangle_2$; with one charge in the first sheet and the other in the second one ($m=1, 4, 7\ldots, -2, -5, -8, \ldots$), we obtain $\langle\psi\bar{\psi}\rangle_2$ or $\langle\bar{\psi}\psi\rangle_3$; and taking the limit with one charge in the first sheet and the other in the third one ($m=2, 5, 8\ldots, -1, -4, -7\ldots$), we get $\langle\psi\bar{\psi}\rangle_3$ or $\langle\bar{\psi}\psi\rangle_1$.

Let us now observe that the spin of the field ψ is closely related to the topology of certain spaces. Consider the curves connecting the two charges before we take the limit. In the many sheeted euclidean space we have been considering, with two branch points placed at x_2 and y_2, these curves belong to topologically inequivalent classes, characterized by the number m.

Therefore, we cannot associate a certain ordering of the correlation functions with a unique topological class.

Consider now a two-dimensional space with a circular hole with diammetrically opposite points identified. If we place the two charges in this space, the curves connecting them, belong to only two topological classes characterized by m even and m odd respectively (m is the number of times a certain curve crosses the hole). This hole is a deformation of the cut along the string so we see that for fermions, the two possible orderings of the correlation function are associated with topologically inequivalent classes of this space.

For spin-$\frac{1}{3}$, for instance, we must consider a space with a circular hole in which every three points placed at 120° from each other are identified. This space has three topological classes, each one corresponding to a certain sheet of the $s=\frac{1}{3}$ Schwinger functions.

In this way, for every value of the spin we way construct a space whose topology is associated with the various orderings of the correlation functions. This observation suggests a topological interpretation for the statistics of a field.

Let us finally remark that constructing the disorder–disorder correlation function for the massive Thirring model along the lines described in eqs. (2.11)–(2.13), we would be able to show that due to the cosine term, path independence implies quantization of b. This agrees with what is known from an analysis of the operator solution [12].

3. Massless Schwinger model

The bosonized form of the Schwinger model lagrangian density in the euclidean region is

$$\mathscr{L}_{\mathrm{Sch}} = -\tfrac{1}{2}\phi\,\partial^2\phi + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{ie}{\sqrt{\pi}}\,\varepsilon^{\mu\nu}\,\partial_\nu\phi A_\mu + \frac{ie\theta}{2\pi}\,\varepsilon^{\mu\nu}\,\partial_\nu A_\mu\,, \tag{3.1}$$

where ϕ is the pseudopotential and the last term exhibits the θ dependence of the theory.

In the functional framework we shall integrate the negative exponential of (3.1) with the external leg insertions over ϕ, and the transversal A_μ.

Notice that contrary to the conventional Grassmann formulation, the current that couples to the electromagnetic field is identically conserved. A gauge tranformation, therefore, leaves the fermionic degrees of freedom unchanged. This means that in this framework we will automatically compute the gauge invariant correlation functions. For instance, we have

$$\begin{aligned}\langle T(x,y)\rangle &\equiv \left\langle \psi(x)\exp\left\{-ie\int_{x,C}^{y} A_\mu(\mathfrak{z})\,\mathrm{d}\mathfrak{z}_\mu\right\}\bar\psi(y)\right\rangle \\ &= N\int [\mathrm{D}A_\mu^{\mathrm{T}}][\mathrm{D}\phi]\,\mathrm{e}^{-S_{\mathrm{Sch}}} \\ &\quad\times\exp\left\{i\sqrt{\pi}\left[\gamma_x^5\phi(x) - i\int_{x,C}^{y}\varepsilon^{\mu\nu}\,\partial_\nu\phi(z)\,\mathrm{d}z_\mu + \gamma_y^5\phi(y)\right]\right\},\end{aligned} \tag{3.2}$$

where the coefficients a and b corresponding to the fermionic degrees of freedom have been fixed to be both equal to $\sqrt{\pi}$, since the Schwinger model couples the electromagnetic field to a canonical fermion ($a = b = \sqrt{\pi}$).

Integrating first over A_μ^{T}, all the possible windings of the electromagnetic field are included, since the electromagnetic action is finite for any topological charge. Therefore, in this case, (3.2) should describe the θ-vacua correlation function:

$$\begin{aligned}\langle\theta|T(x,y)|\theta\rangle &= N\int[\mathrm{D}\phi]\exp\left[-\int \mathrm{d}^2z\left\{\tfrac{1}{2}\phi\left(-\partial^2+\frac{e^2}{\pi}\right)\phi + \phi[\alpha+\beta]\right\}\right] \\ &\quad\times\exp\{\tfrac{1}{2}i\theta[\gamma_x^5+\gamma_y^5]\}\,,\end{aligned} \tag{3.3}$$

where α and β are given by (2.5) and (2.12) respectively.

Note that (3.3) differs from the corresponding equations [(2.5), (2.12) and (2.19)] of the Thirring model by the mass that the ϕ field has acquired. The origin of this mass can be traced back to the fact that the coupling to the A_μ field provides us (going now to a language dual to the one employed in sect. 2) with a background of magnetic sources, so that the A_μ^{T} integration has the result of producing a magnetic plasma. It is immediately apparent that due to this mass term the chiral selection rule will be lost and that the θ vacua can be regarded as a chiral (magnetic monopole) condensate.

The ϕ integration is now saturated by the solutions of the massive Poisson equation yielding the result [15]:

$$\langle\theta|T(x,y)|\theta\rangle = \pm\exp\left\{-\pi\gamma_x^5\gamma_y^5\Delta(y-x) + \tfrac{1}{2}\pi\int_{x,C}^{y} \mathrm{d}\xi_\mu \int_{x,C}^{y} \mathrm{d}\eta_\alpha \varepsilon^{\mu\nu}\varepsilon^{\alpha\beta}\,\partial_\nu^\xi\partial_\beta^\eta\Delta(\xi-\eta)\right.$$
$$\left. + i\pi\int_{x,C}^{y}[\gamma_x^5\varepsilon^{\mu\nu}\,\partial_\nu^\xi\Delta(x-\xi) + \gamma_y^5\varepsilon^{\mu\nu}\,\partial_\nu^\xi\Delta(y-\xi)]\,\mathrm{d}\xi_\mu\right\}\mathrm{e}^{-\pi\Delta(0)}$$
$$\times\exp\{\tfrac{1}{2}i\theta[\gamma_x^5+\gamma_y^5]\}\,. \tag{3.4}$$

The sign ambiguity has the same origin as in the Thirring model and is related to the two possible orderings. $\Delta(z)$ is the massive two-dimensional Green function.

This equation indeed reproduces the unrenormalized gauge invariant 2-point function of the Schwinger model computed by standard methods [16, 24].

Due to the short-range nature of the ϕ potential, the electric field is concentrated in a thin tube along the string. The aphysical string of the Thirring model now becomes a physical confining tube of electric flux whose width is of order $1/e$. The interaction energy, now depends on the shape and size of the string, as is clearly seen from (3.4).

The present description of QED_2 closely parallels the expected behaviour of a 4-dimensional confining theory [11].

The $\sqrt{\pi}$ gauge correlation functions of ref. [10] can be obtained by simply removing the string with the associated flux tube ($\beta=0$ in eq. (3.3)) obtaining thus for the first time a path integral formulation for this non-canonical gauge.

We see that the spurionization of charge of this gauge reflects itself in the total absence of charges (endpoints of magnetic dipole strings) in the corresponding classical system.

Hence, performing the ϕ integration in (3.3) with $\beta=0$, we get the well-known result

$$\langle\theta|\hat{\psi}_\mathrm{R}(x)\hat{\bar{\psi}}_\mathrm{R}(y)|\theta\rangle = \exp\{-\pi\gamma_x^5\gamma_y^5\Delta(y-x)\}\exp\{\tfrac{1}{2}i\theta[\gamma_x^5+\gamma_y^5]\}\,. \tag{3.5}$$

In order to study the tunnelling amplitudes between different n vacua, and to identify the relevant winding numbers of the A_μ field, it is convenient to perform first the integration over fermionic degrees of freedom. Using our method, we reobtain the results of refs. [16, 17] without having to resort to clustering properties and without explicit use of the 't Hooft [18] Atiyah–Singer [19, 20] mechanism.

Doing the ϕ integration, we get

$$\langle n|T(x,y)|0\rangle = \pm\langle\psi_\mathrm{R}(x)\bar{\psi}_\mathrm{R}(y)\rangle_0 N\int[\mathrm{D}A_\mu^{(n)}]\exp\left\{-\int\mathrm{d}^2z\left[\tfrac{1}{2}A_\mu^\mathrm{T}\left(-\partial^2+\frac{e^2}{\pi}\right)A_\mu^\mathrm{T}\right.\right.$$
$$-ieA_\mu^\mathrm{T}\left[-i\varepsilon^{\mu\lambda}\,\partial_\lambda[\gamma_x^5D(z-x)+\gamma_y^5D(z-y)]\right.$$
$$\left.\left.\left.+\frac{1}{2\pi}\varepsilon^{\mu\lambda}\,\partial_\lambda[\arg(z-x)-\arg(z-y)]\right]\right]\right\}\mathrm{e}^{in\theta}\,, \tag{3.6}$$

where we now restrict ourselves to configurations with winding number n and $\langle \psi_R(x)\bar{\psi}_R(y)\rangle_0$ is given by (2.20) (with $a = b = \sqrt{\pi}$).

Observe that the only configurations that will contribute in (3.6) are the ones whose Chern number n are equal to half of the chirality carried by the operator $T(x, y)$, in our example either 0, ± 2.

Using the fact that asymptotically $A_\mu^{(n)}$ behaves like $(2\pi n/e)\varepsilon^{\mu\nu}\,\partial_\nu D(z)$, we might collect the long distance diverging contributions in the multiplicative factor

$$\exp\{\tfrac{1}{2}\pi D(\infty)[2n - (\gamma_x^5 + \gamma_y^5)]^2\}, \tag{3.7}$$

which corresponds to the realization of the 't Hooft–Atiyah–Singer mechanism in this formulation.

Computing (3.6) with the correct winding number we reobtain eq. (3.4).

Again, by dropping the string and performing first the ϕ integration, we are led to the $\sqrt{\pi}$ gauge tunnelling amplitudes. It is straightforward that in this case the relevant winding number contribution to the one-point function $\langle \hat{\psi}(x)\rangle$ is $\pm\frac{1}{2}$. These configurations which are known to be related to the confining properties of the Schwinger model [21, 24] are not compactifiable, and the fact that this one-point function is non-vanishing had not yet been understood within the traditional path integral framework. Nevertheless, our results suggest that there is some generalization of 't Hooft–Atiyah–Singer mechanism for such configurations.

By reinterpreting the right-hand side of (3.6) as being defined in the many-sheeted euclidean space, we can obtain the gauge dependent tunnelling amplitudes $\langle n|\psi(x)\bar{\psi}(y)|0\rangle$ in the Landau gauge. This reinterpretation amounts to dropping the δ term in the Cauchy–Riemann equation (2.18) which will eliminate the string and lead us directly to the expression obtained in ref. [22] in the traditional way. The functional integral now becomes path independent, up to the ambiguity in sign.

In all cases considered here, we can obtain the more general correlation functions by simply introducing more charges (strings) and magnetic monopoles and computing the exponential of the corresponding classical interaction energy.

The previous discussion could be generalized to the coupling of the electromagnetic field to a generic Thirring field (a and b arbitrary). The main novel feature would be that the relevant winding number of the A_μ field would be integer multiples of the spin of the Thirring field and therefore in the generic case, non-compactifiable [15].

All that we said about many-sheetedness orderings and path topology in sect. 2 is also true here.

4. Conclusions

Based in the order-disorder structure of two-dimensional bosonized fields, we have given a general functional integral formulation for fields with an arbitrary spin. Of course, if we have Fermi or Bose statistics, our formulation reproduces the results obtained by using standard methods. Nevertheless, it gives a new insight into those problems.

In the Thirring model, for instance, we have been able to obtain with our functional integral method, the general Klaiber solution. Such a formulation will certainly be needed in other models where general statistics fields play a role. For instance we expect it to be useful in obtaining Green functions of the chiral Gross–Neveu model, and possibly also in the investigation of the scaling limit of $Z(N)$ generalizations of the Ising model.

On the other hand in the Schwinger model we reobtain well-known results from a different point of view. We were also able to compute *via* path integrals the correlation functions in a non-canonical gauge like the $\sqrt{\pi}$ gauge, exhibiting a mechanism that seems to go beyond the classical 't Hooft–Atiyah–Singer one.

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